Матрично-векторное дифф.

Выпуклость

сопряженные мн-ва

супремум

инфимум

Conjugate (dual) function

$\mathrm{dom}\, f \subseteq E$ $\qquad$ $\mathrm{dom}\, f^*_y \subseteq E^*$

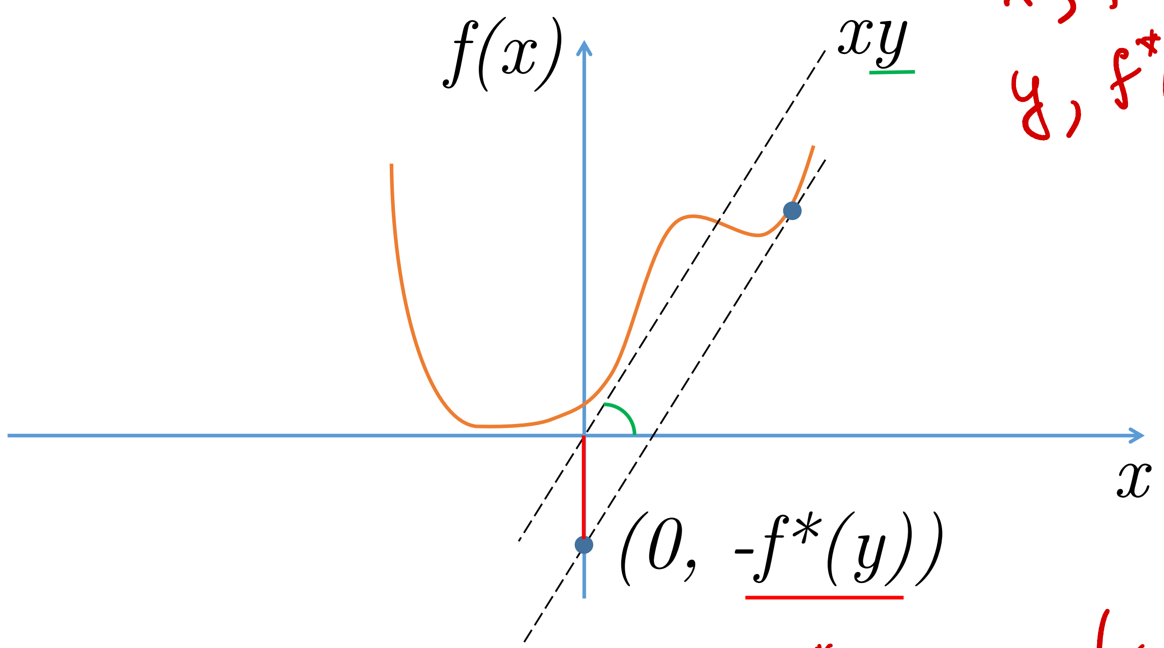
Let $f : \mathbb{R}^n \to \mathbb{R}$. The function $f^* : \mathbb{R}^n \to \mathbb{R}$ is called convex conjugate (Fenchel's conjugate, dual) $f(x)$ and is defined as follows:

$$\boxed{f^*(y) = \sup_{x \in \mathbf{dom}\ f} (\langle y, x\rangle - f(x)).} = \sup_{x \in \mathrm{dom} f} \varphi(x, y)$$

Let's notice, that the domain of the function f^* is the set of those y, where the supremum is finite.

$x,\ f(x)$

$y,\ f^*(y)$



$$f^*(y) = \sup_{x \in \mathrm{dom} f} \left(\langle x, y\rangle - f(x) \right)$$

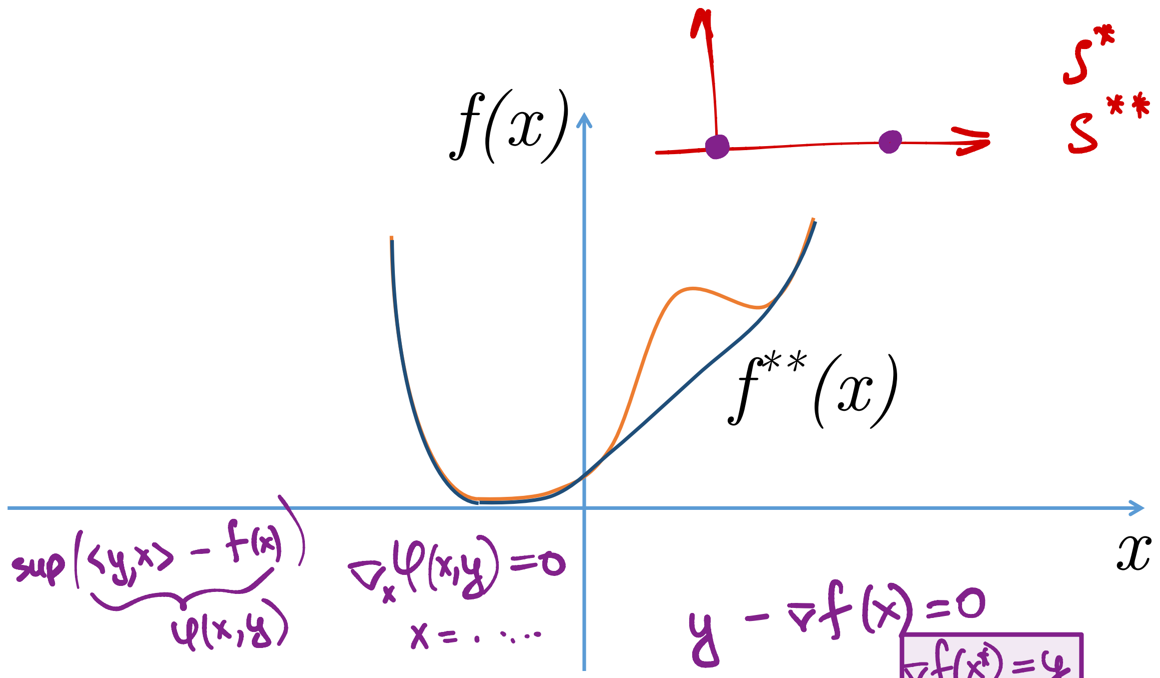
Properties

замкнутая

- $f^*(y)$ - always closed convex function (point-wise supremum of closed convex functions) on y (Function $f : X \to R$ is called closed if $\mathbf{epi}(f)$ is a closed set in $X \times R$)
- Fenchel–Young inequality:

$$\boxed{f(x) + f^*(y) \geq \langle y, x\rangle}$$

- Let the functions $f(x), f^*(y), f^{**}(x)$ are defined on the $\mathbb{R}^n$. Then $\boxed{f^{**}(x) = f(x)}$ if and only if $f(x)$ - proper convex function (Fenchel - Moreau theorem). (proper convex function = closed convex function)
- Consequence from Fenchel–Young inequality: $\boxed{f(x) \geq f^{**}(x)}$



- The Legendre transformation as a special case of Fenchel's conjugate (in case of differentiable function). Let $f(x)$ - convex and differentiable, $\mathbf{dom}\ f = \mathbb{R}^n$. Then $x^* = \underset{x}{\text{argmin}} \langle x, y\rangle - f(x)$. In that case $y = \nabla f(x^*)$. That's why:

$$f^*(y) = \langle \nabla f(x^*), x^* \rangle - f(x^*)$$

$$f^*(y) = \langle \nabla f(z), z \rangle - f(z), \qquad y = \nabla f(z), \quad z \in \mathbb{R}^n$$

- Let $f(x, y) = f_1(x) + f_2(y)$, where f_1, f_2 - convex functions, then

$$\boxed{f^*(p, q) = f_1^*(p) + f_2^*(q)}$$

- Let $\boxed{f(x) \leq g(x)}$ $\forall x \in X$. Let also $f^*(y), g^*(y)$ are defined on Y. Then $\forall x \in X, \forall y \in Y$

$$\boxed{f^*(y) \geq g^*(y)} \quad \boxed{f^{**}(x) \leq g^{**}(x)}$$

Examples

The scheme of recovering the convex conjugate is pretty algorithmic:

1. Write down the definition $f^*(y) = \sup\limits_{x \in \mathbf{dom}\ f} (\langle y, x\rangle - f(x)) = \sup\limits_{x \in \mathbf{dom}\ f} f(x, y)$
2. Find those y, where $\sup\limits_{x \in \mathbf{dom}\ f} f(x, y)$ is finite. That's the domain of the dual function $f^*(y)$
3. Find x^*, which maximize $f(x, y)$ as a function on x. $f^*(y) = f(x^*, y)$

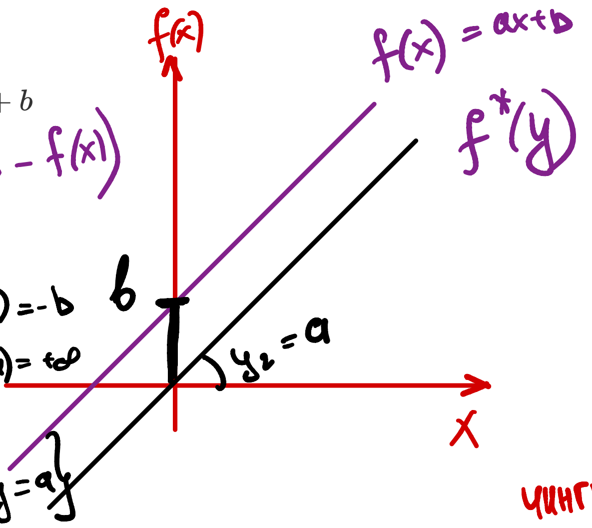
Example 1

Find $f^*(y)$, if $f(x) = ax + b$

$$f^*(y) = \sup_{x \in \mathrm{dom} f} (\langle x, y\rangle - f(x))$$

$$f^*(y) = \begin{cases} y = a, & f^*(a) = -b \\ y \neq a, & f^*(a) = +\infty \end{cases}$$

$$\mathrm{dom}\, f^* = \{y \in \mathbb{R} \mid y = a\}$$



чингиз$(x) = y^T x$

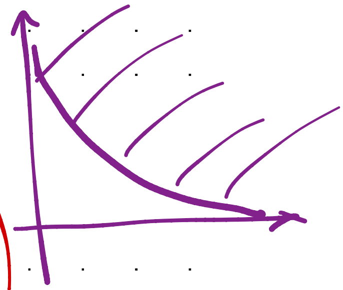
$f(x) = ax + b$

$$f^*(y) = \sup_{x \in \mathrm{dom} f} \left(\langle x, y \rangle - f(x) \right) =$$

$$= \sup_{x \in \mathbb{R}} \left(xy - ax - b \right) = \sup_{x \in \mathbb{R}} \left((y-a)x - b \right)$$

$y \neq a$

$y = a \quad f^*(y) = -b$



Example 2

Find $f^*(y)$, if $f(x) = \dfrac{1}{x}, x \in \mathbb{R}_{++}$

$$f^*(y) = \sup_{x \in \mathbb{R}^{++}} \left(xy - f(x) \right) = \sup_{x \in \mathbb{R}^{++}} \left(\underbrace{xy - \frac{1}{x}}_{\varphi(x,y)} \right)$$

$$\nabla_x \varphi(x,y) = 0 \Rightarrow y + \frac{1}{x^2} = 0$$

$$\boxed{\frac{1}{x^2} = -y}$$

$y \leq 0$

$$x^* = \frac{1}{\sqrt{-y}}$$

$$f^*(y) = \varphi(x^*, y) = \frac{1}{\sqrt{-y}} \cdot y - \frac{1}{\frac{1}{\sqrt{-y}}} = -\sqrt{-y} - \sqrt{-y} = -2\sqrt{-y}.$$

$$\sqrt{-y} \cdot \sqrt{-y} = -y$$

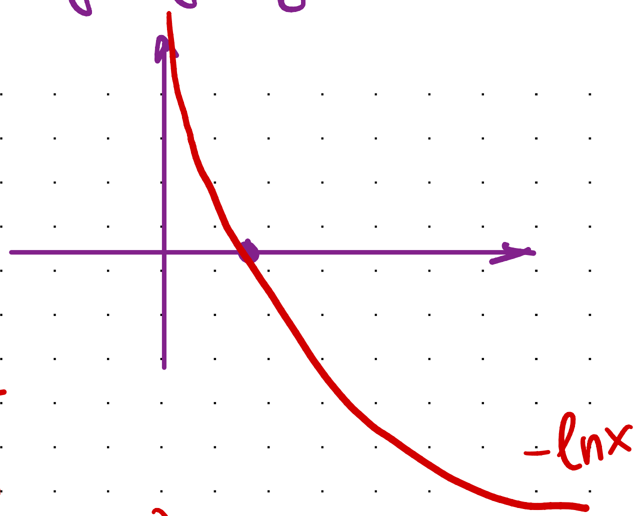
Ответ: $\boxed{f^*(y) = -2\sqrt{-y}, \; y \leq 0}$

Example 3

Find $f^*(y)$, if $f(x) = -\log x, \quad x \in \mathbb{R}_{++}$

$f(x) = -\ln x \quad , x > 0$

$$f^*(y) = \sup_{x \in \mathrm{dom} f}$$



$$y + \frac{1}{x} = 0$$

$$x^* = -\frac{1}{y}$$

$$f^*(y) = -1 + \ln\left(-\frac{1}{y}\right)$$

$y < 0$

$$\Rightarrow \boxed{f^*(y) = -1 - \ln(-y)}$$

Example 4

Find $f^*(y)$, if $f(x) = e^x$

Example 5

Find $f^*(y)$, if $f(x) = x \log x, x \neq 0, \quad f(0) = 0, \quad x \in \mathbb{R}_+$

Example 6

Find $f^*(y)$, if $f(x) = \frac{1}{2} x^T A x, \quad A \in \mathbb{S}^n_{++}$

Example 7

Revenue and profit functions. We consider a business or enterprise that consumes n resources and produces a product that can be sold. We let $r = (r_1, \dots, r_n)$ denote the vector of resource quantities consumed, and $S(r)$ denote the sales revenue derived from the product produced (as a function of the resources consumed). Now let p_i denote the price (per unit) of resource i, so the total amount paid for resources by the enterprise is $p^\top r$. The profit derived by the firm is then $S(r) - p^\top r$. Let us fix the prices of the resources, and ask what is the maximum profit that can be made, by wisely choosing the quantities of resources consumed. This maximum profit is given by

$$M(p) = \sup_r \left(S(r) - p^\top r\right)$$

The function $M(p)$ gives the maximum profit attainable, as a function of the resource prices. In terms of conjugate functions, we can express M as

$$M(p) = (-S)^*(-p).$$

Thus the maximum profit (as a function of resource prices) is closely related to the conjugate of gross sales (as a function of resources consumed).

Example 8

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, with dual norm $\|\cdot\|_*$. Show that the conjugate of $f(x) = \|x\|$ is:

$$f^*(y) = \begin{cases} 0, & \|y\|_* \le 1 \\ \infty, & \text{otherwise} \end{cases}$$

Dual norm

Let $\|x\|$ be the norm in the primal space $x \in S \subseteq \mathbb{R}^n$, then the following expression defines dual norm:

$$\boxed{\|x\|_* = \sup_{\|y\| \le 1} \langle y, x \rangle}$$

$\|x\|$

сопряженная норма

$$\|y\|_* = \sup_{\|x\| \le 1} \langle y, x \rangle$$

пример

$y = 1000 \cdot 1^n$

$$\|y\|_* = \sup_{\|x\| \le 1} 1000 \cdot 1^T x$$

n

1000

$x \in \mathbb{R}^n \quad f(x) = \|x\| \qquad f^*(y) \quad y \in \mathbb{R}^n$

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \left(x^T y - \|x\| \right)$$

ФАКТ
КБШ: $\boxed{x^T y \leq \|x\| \cdot \|y\|_*}$

$$x^T y - \|x\| \leq \|x\| \cdot \|y\|_* - \|x\|$$

$$x^T y - \|x\| \leq \|x\| \cdot \left(\|y\|_* - 1 \right)$$

- если $\|y\|_* > 1$, то $\sup \varphi(x, y) = +\infty$
- если $\|y\|_* \leq 1$, то $f^*(y) = 0$.

$$\boxed{\text{Ответ: } f^*(y) = 0, \quad \|y\|_* \leq 1}$$

$(\|x\|_\infty)_* = \|x\|_1$

The intuition for the finite-dimension space is how the linear function (element of the dual space) $f_y(\cdot)$ could stretch the elements of the primal space with respect to their size, i.e. $\|y\|_* = \sup\limits_{x \neq 0} \dfrac{\langle y, x\rangle}{\|x\|}$

$p = 1$ $\quad (\|x\|_1)_* = \|x\|_\infty$ $\qquad$ $p, q \geq 1$ $\quad (\|x\|_p)_* = \|x\|_q$

Properties

- One can easily define the dual norm as:

$$
\|x\|_* = \sup_{y \neq 0} \frac{\langle y, x\rangle}{\|y\|}
$$

$\boxed{\frac{1}{p} + \frac{1}{q} = 1}$

- The dual norm is also a norm itself
- For any $x \in E, y \in E^*$: $\boxed{x^\top y \leq \|x\| \cdot \|y\|_*}$
- $(\|x\|_p)_* = \|x\|_q$ if $\dfrac{1}{p} + \dfrac{1}{q} = 1$, where $p, q \geq 1$

$\|x\|_2$

$(\|x\|_2)_* = \|x\|_2$

Examples

- Let $f(x) = \|x\|$, then $f^*(y) = \mathbb{O}_{\|y\|_* \leq 1}$
- The Euclidian norm is self dual $(\|x\|_2)_* = \|x\|_2$.

Materials

- Convex Optimization materials by Boyd and Vandenberghe.
- Методы оптимизации, Часть I. Введение в выпуклый анализ и теорию оптимизации. Жадан ВГ.