проект + д/з + тесты
40% 30% 30%

если General formulation

экз ↘ уд 3 → вес экз $\frac{3}{4}$

↗ уд 4 вес сем. $\frac{1}{4}$

$$
\boxed{\begin{aligned}
&\min_{x \in \mathbb{R}^n} f(x) \\
\text{s.t. } & g_i(x) \leq 0, \; i = 1, \ldots, m \\
& h_j(x) = 0, \; j = 1, \ldots, k
\end{aligned}}
$$

+ бонусы
коммиты
fmin.xyz

+ лекц. к/р - (вспец ...)

Some necessary or/and sufficient conditions are known (See Optimality conditions. KKT and Convex optimization problem)

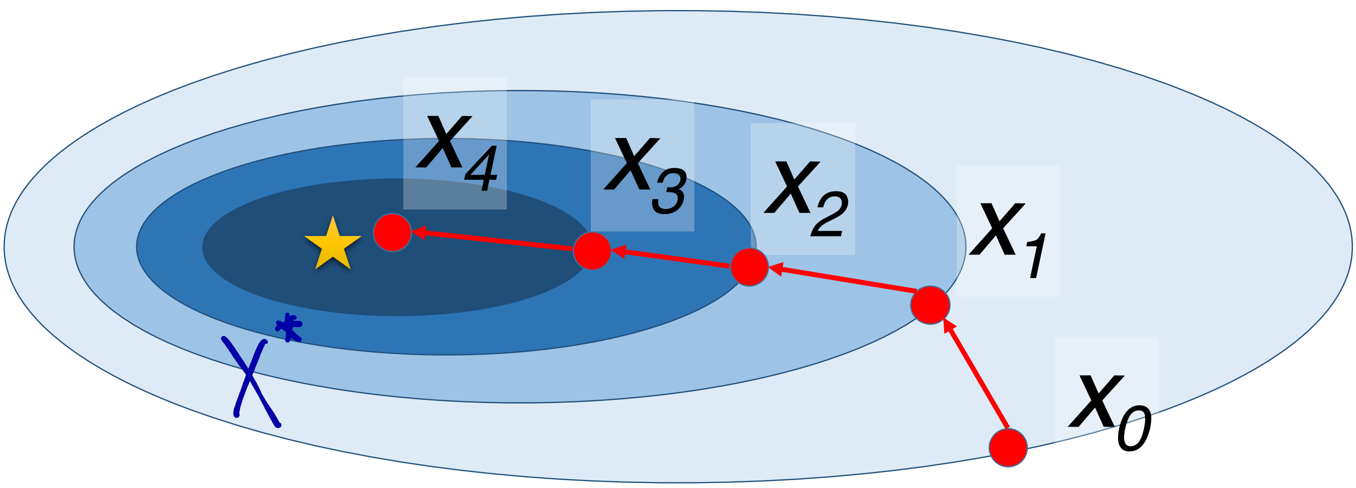
- In fact, there might be very challenging to recognize the convenient form of optimization problem.
- Analytical solution of KKT could be inviable.

Iterative methods

Typically, the methods generate an infinite sequence of approximate solutions

$$\{x_t\},$$

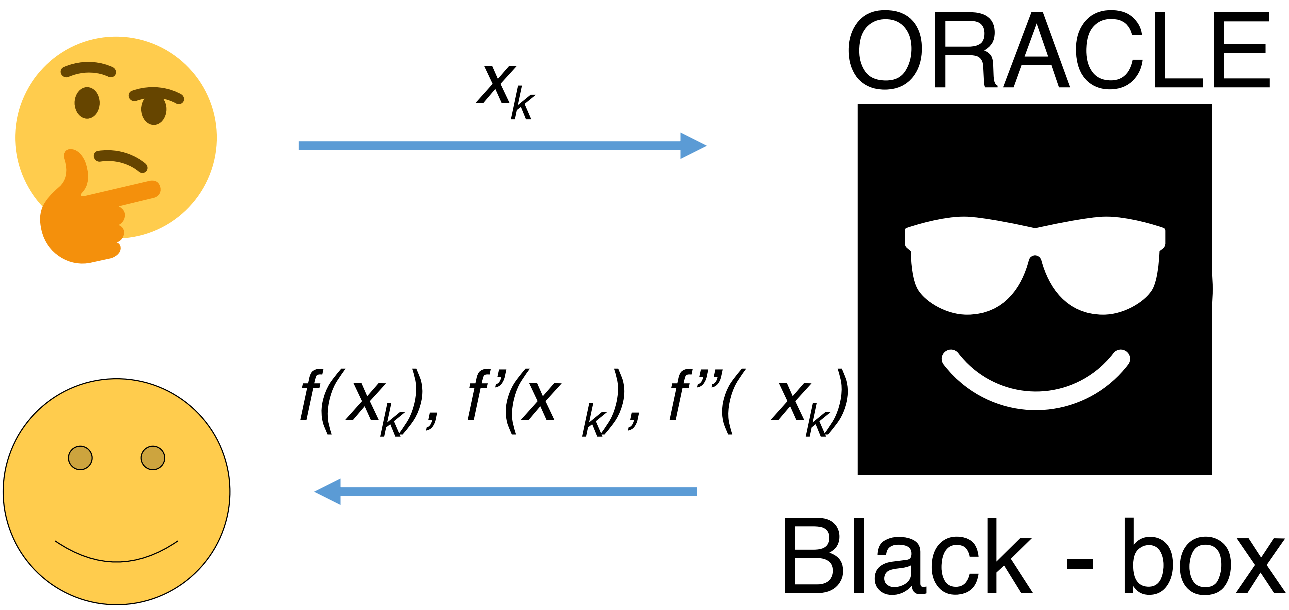
which for a finite number of steps (or better - time) converges to an optimal (at least one of the optimal) solution x_*.




```
def GeneralScheme(x, epsilon):
    while not StopCriterion(x, epsilon):
        OracleResponse = RequestOracle(x)
        x = NextPoint(x, OracleResponse)
    return x
```

Oracle conception

$$x_{k+1} = \underline{x_k - \alpha_k \nabla f(x_k)}$$



$f: \mathbb{R}^n \to \mathbb{R}$

$\min\limits_{x \in \mathbb{R}^n} f(x)$

+ задана L:

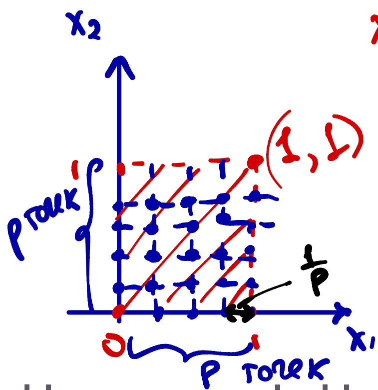
$|f(x) - f(y)| \leq L \cdot \|x - y\| \quad \forall x, y \in \mathbb{R}^n$

+ задано S:

$S = \mathbb{B}^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1\}$

Complexity

Challenges



Unsolvability

In general, **optimization problems are unsolvable.** ¯\\(ツ)/¯

Consider the following simple optimization problem of a function over unit cube:

$$\min_{x \in \mathbb{R}^n} f(x)$$
$$\text{s.t. } x \in \mathbb{B}^n$$

We assume, that the objective function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{B}^n$:

$$\boxed{|f(x) - f(y)| \leq L\|x - y\|_\infty} \forall x, y \in \mathbb{B}^n,$$

with some constant L (Lipschitz constant). Here $\mathbb{B}^n$ - the n-dimensional unit cube

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, i = 1, \dots, n\}$$

Our goal is to find such $\tilde{x}$: $|f(\tilde{x}) - f^*| \leq \varepsilon$ for some positive ε. Here f^* is the global minima of the problem. Uniform grid with p points on each dimension guarantees at least this quality:

$$\boxed{\|\tilde{x} - x_*\|_\infty \leq \frac{1}{2p},}$$

which means, that

$$\boxed{|f(\tilde{x}) - f(x_*)| \leq \frac{L}{2p}} \leq \varepsilon$$

Всего p^n $(p+1)^n$

$\frac{L}{2p} = \varepsilon$

$p = \frac{L}{2\varepsilon}$

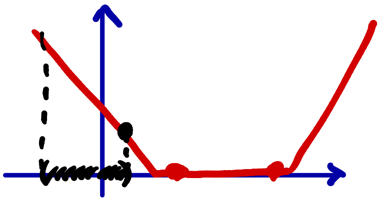
Our goal is to find the p for some ε. So, we need to sample $\left(\frac{L}{2\varepsilon}\right)^n$ points, since we need to measure function in p^n points. Doesn't look scary, but if we'll take $L = 2, n = 11, \varepsilon = 0.01$, computations on the modern personal computers will take 31,250,000 years.

1 Неразрешимость мн. зад. опт.

нужно $\left(\frac{L}{2\varepsilon}\right)^n$

$L = 2$
$\varepsilon = 0.01$
$n = 11$

2 Stopping rules



- Argument closeness:

$$\boxed{\|x_k - x_*\|_2 < \varepsilon}$$

- Function value closeness:

$f_k = f(x_k)$

$$\boxed{\|f_k - f^*\|_2 < \varepsilon}$$

- Closeness to a critical point

$$\boxed{\|f'(x_k)\|_2 < \varepsilon}$$

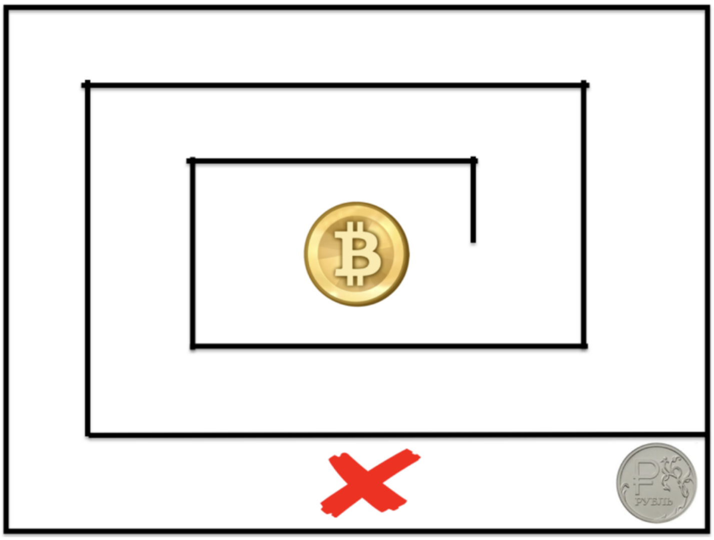
But x_* and $f^* = f(x_*)$ are unknown!

Sometimes, we can use the trick: умный ход

$$\boxed{\|x_{k+1} - x_k\|} = \|x_{k+1} - x_k + x_* - x_*\| \leq \underbrace{\|x_{k+1} - x_*\|}_{\leq \varepsilon} + \underbrace{\|x_k - x_*\|}_{\leq \varepsilon} \leq 2\varepsilon$$

Note: it's better to use relative changing of these values, i.e. $\frac{\|x_{k+1} - x_k\|_2}{\|x_k\|_2}$.

Local nature of the methods



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Скорость сходимости $r_k = |f_k - f^*|$

Пусть r_k — посл. чисел ≥ 0, сх.-ся к 0. $r_k = \|x_k - x^*\|$

<u>Опр.</u> Линейная сходимость.

посл. r_k — сходится линейно, если

$$\boxed{r_k \leq c \cdot q^k} \quad \forall k$$

$c > 0$, $0 < q < 1$

идея: $\|x_{k+1} - x^*\| \leq q \cdot \|x_k - x^*\|$

$\|x_{k+1} - x^*\| \leq q^2 \cdot \|x_{k-1} - x^*\|$

так же наз.

- экспоненциальная сх-ть
- геометрическая сх-ть

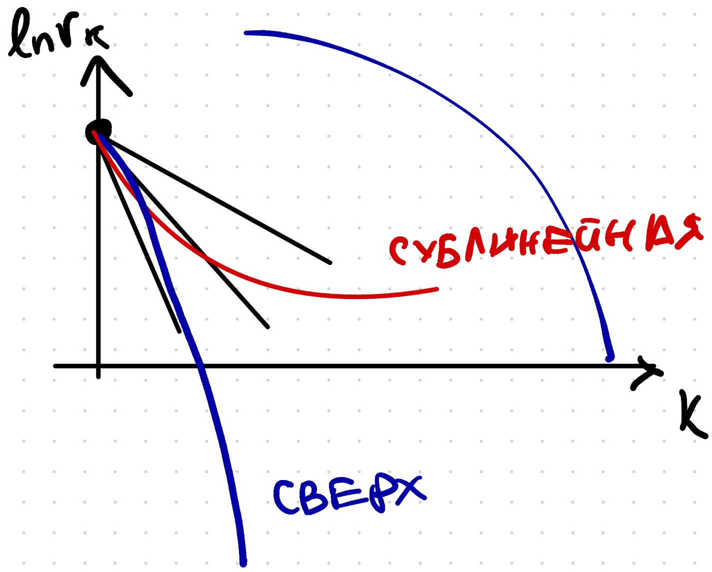
пример: $r_k = \frac{1}{3^k}$ $\quad c = 1$, $q = \frac{1}{3}$

$r_k = \frac{4}{3^k}$ $\quad$ скорость сх-ть

$q = ? = \frac{1}{3}$

если r_k сх-ся быстрее, чем линейная,
то сходимость – сверхлинейная

пример



если r_k сх-ся медленнее, чем
$\forall$ линейная сх-ть $\longrightarrow$ сублинейная

<u>пример</u>:

гарм.

$$r_k = \left\{ \frac{1}{k} \right\}_{k=1}^{\infty}$$

$$S_n = \sum_{i=1}^{n} r_i$$

Как определить сходимость?

① Тест корней

$$q = \limsup_{k\to\infty} \left(r_k\right)^{\frac{1}{k}}$$

- $0 \leqslant q < 1$ – сходится линейно со скоростью q
- если $q = 0$ – сходится сверхлинейно
- если $q = 1$ – сходится сублинейно

$q > 1$ невозможно

② Тест отношений (RATIO TEST)

$$q = \lim_{k\to\infty} \frac{r_{k+1}}{r_k}$$

- $0 \leq q < 1$ – посл. сх-ся линейно со скоростью q
- $q = 0$ – сверхлинейная сх-ть
- $q = 1$ – сублинейная сх-ть.

• если $\nexists\, q$ если $q := \lim\limits_{k\to\infty} \sup \frac{r_{k+1}}{r_k} < 1$

можно сказать, что r_k сх-ся линейно со скоростью не выше q

• если $\nexists\, q$ $q := \lim\limits_{k\to\infty} \inf \frac{r_{k+1}}{r_k} = 1$

$\Rightarrow$ r_k сх-тся сублинейно

Пример:

$$r_k = \frac{1}{k} \qquad \lim_{k\to\infty} \frac{r_{k+1}}{r_k} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

пример:

$$r_k = \frac{1}{k^2} \qquad \frac{k^2}{(k+1)^2}$$

сх-ся сублин.

$$r_k = \frac{1}{k^q} \qquad q > 1$$

сублинейно

задача:

$$r_k = \left(\frac{1}{k}\right)^k \qquad \lim r_k^{\frac{1}{k}}$$

$$\lim_{k\to\infty} \left(\left(\frac{1}{k}\right)^k\right)^{\frac{1}{k}} = \frac{1}{k} = 0$$

сх-ть сверхлинейная

пример:
сверхлинейной
сх-ти:

$c > 0$

Квадратичная сходимость:

$$\boxed{r_k \le c \cdot q^{2^k}}$$

$0 < q < 1$

пример: $r_k = (0.707)^k$

лин. сх-ть

$q = 0.707$

$$r_k = (0.707)^{2^k}$$

$$r_k = 1 + (0.5)^{2^k}$$

$$\tilde{r}_k = r_k - 1 = (0.5)^{2^k}$$

$$r_k = \frac{1}{k!} \quad ??$$